Lecture 16

01-11-18.

§ homogeneous. Linear equation with constant coefficient.

Def: • A system of 1^{st} order Linear differential equation with constant coefficient is of the form

$$\vec{y}'(t) = A\vec{y}(t) + \vec{r}(t).$$

• it is called homogeneous if we have $\vec{r}(t) = 0$.
which is of the form $\vec{y}'(t) = Ay(t)$. ----(*).
if $\vec{y}_1, \dots, \vec{y}_n$ fundamental set of sol for *

$$\Phi(t) = \begin{pmatrix} | & & | \\ y_1, & \cdots & y_n \\ | & & | \end{pmatrix}$$ is called the fundamental matrix

Rk: • We prefer to work over $\mathbb{C}$ first even if we interested in $\mathbb{R}$-valued solution for $A \in M_{n\times n}(\mathbb{R})$.
• It is because we have a better theory of diagonalizing a matrix.

idea: • If A is a 1×1 matrix, we have
$$y' = Ay \Rightarrow y = e^{At}$$

• Now, we try to see if this formula holds in general

For matrix:

$$e^{At} := 1 + (At) + \frac{(At)^2}{2!} + \cdots + \frac{(At)^k}{k!} + \cdots\cdots$$

Formal computation:

$$\frac{de^{(At)}}{dt} = \frac{d}{dt}\left(id + tA + \frac{t^2A^2}{2!} + \frac{t^3A^3}{3!} + \cdots \right)$$

$$= A + tA^2 + \frac{t^2A^3}{2!} + \cdots$$

$$= A\left(id + tA + \frac{t^2A^2}{2!} + \cdots\right) = Ae^{At}.$$

i.e. "e^{At}" is a candiate for the fundamental matrix

E.g. $A \in M_{n\times n}(\mathbb{R})$, there are three cases about eigenvalues

i) distinct real eigenvalues $\lambda_1 \neq \lambda_2$

ii) " complex " $\lambda_1 = \alpha + i\mu$, $\lambda_2 = \alpha - i\mu$.

iii) repeated real eigenvalue $\lambda_1 = \lambda_2 = \lambda$.

1) geometric mult. of λ = alg. mult. of λ

2) geometric " of λ < alg mult. of λ.

Case i) $\exists\, Q = \begin{pmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{pmatrix}$ $\mathbb{R}$-valued matrix

eigenvector of λ_1

eigenvector of λ_2.

s.t. $Q^{-1}AQ = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = D.$

in this case we write $A = QDQ^{-1}$

Hence $\vec{y}\,'(t) = (QDQ^{-1})\vec{y}(t).$

Therefore, if we let $\vec{z}(t) = Q^{-1}\vec{y}(t)$.

we reduce the equation to $\vec{z}'(t) = D\,\vec{z}(t)$.

$$\longrightarrow \begin{pmatrix} z_1'(t) \\ z_2'(t) \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

Fundamental matrix $Z(t) = e^{\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} t}$

$$= id + (Dt) + \frac{t^2 D^2}{2!} + \cdots + \frac{t^k D^k}{k!} + \cdots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \lambda_1 t & 0 \\ 0 & \lambda_2 t \end{pmatrix} + \begin{pmatrix} \frac{\lambda_1^2 t^2}{2!} & 0 \\ 0 & \frac{\lambda_2^2 t^2}{2!} \end{pmatrix} + \cdots + \begin{pmatrix} \frac{\lambda_1^k t^k}{k!} & 0 \\ 0 & \frac{\lambda_2^k t^k}{k!} \end{pmatrix} + \cdots = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}.$$

Back to the original equation:

$$\Phi = Q \cdot Z = \begin{pmatrix} e^{\lambda_1 t} v_1, & e^{\lambda_2 t} v_2 \end{pmatrix}.$$

i.e. we solve for $\vec{y}_1 = e^{\lambda_1 t}\begin{pmatrix} v_1 \end{pmatrix}$, $\vec{y}_2 = e^{\lambda_2 t}\begin{pmatrix} v_2 \end{pmatrix}$.

at $t=0$: $\Phi = \begin{pmatrix} v_1 & v_2 \end{pmatrix} \rightsquigarrow \det(\Phi) \neq 0$.

i.e. Φ fundamental.

ii) $\lambda_1 = \alpha + i\mu$ ($\mu \neq 0$), $\lambda_2 = \alpha - i\mu$, then $\Phi = (e^{\lambda_1 t} v_1, e^{\lambda_2 t} v_2)$. $v_2 = \overline{v_1}$.

$\therefore \vec{y}_1 = e^{\lambda_1 t} v_1$, $\vec{y}_2 = e^{\overline{\lambda_1} t}\, \overline{v_1}$.

We take a new $\vec{x}_1 = \mathrm{Re}(e^{\lambda t} v_1)$

$\vec{x}_2 = \mathrm{Im}(e^{\lambda t} v_1)$

$\rightsquigarrow \quad \Phi = \begin{pmatrix} | & | \\ \vec{x}_1 & \vec{x}_2 \\ | & | \end{pmatrix}$ is a $\mathbb{R}$-valued fundamental matrix.

iii) <u>1)</u> $\lambda_1 = \lambda_2 = \lambda$ $\mathbb{R}$-valued with geometric multiplicity of λ = alg. mult. of λ.

$\rightsquigarrow \quad \exists\, Q = \left(\boxed{\begin{matrix} | \\ v_1 \\ | \end{matrix}} \; \boxed{\begin{matrix} | \\ v_2 \\ | \end{matrix}} \right)$ $\mathbb{R}$-valued matrix. ← eigenvectors of λ.

and we have $\Phi = \begin{pmatrix} | & | \\ e^{\lambda t} v_1 , & e^{\lambda t} v_2 \\ | & | \end{pmatrix}$.

same as before

<u>2)</u> $\lambda_1 = \lambda_2 = \lambda$ with geometric mult. $= 1$.

$<$ algebraic mult

<u>In that Case:</u> We can find $Q = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}$ $\mathbb{C}$-valued s.t.

- $Q^{-1} A Q = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ ← Jordan normal form.

<u>i.e.</u> $A v_1 = \lambda v_1$, $A v_2 = \lambda v_2 + v_1$.

↑ eigenvector ↑ generalized eigenvector.

- Indeed we can have Q being $\mathbb{R}$-valued if we have both λ & A $\mathbb{R}$-valued.

Now: Let $\vec{z} = Q^{-1}\vec{y}$ and consider $\vec{z}' = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \vec{z}$ be the equation.

Let's write $D = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ and $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

then we have: $N^2 = 0$, $DN = ND = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}$

then we compute:

$$e^{t(D+N)} = id + t(D+N) + \frac{t^2}{2!}(D+N)^2 + \cdots + \frac{t^k (D+N)^k}{k!} + \cdots$$

Now since $DN = ND$, we can compute $(D+N)^k$ use binomial

$$(D+N)^k = D^k + kD^{k-1}N.$$

$$e^{t(D+N)} = id + tD + \frac{t^2 D^2}{2!} + \cdots + \frac{t^k D^k}{k!} + \cdots$$

$$+ tN\left(id + tD + \frac{t^2 D^2}{2!} + \cdots\right).$$

$$= \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{pmatrix} + tN \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}.$$

Therefore the fundamental matrix for original equation $:=$

$$\Phi = Q e^{t(D+N)} = \begin{pmatrix} | & | \\ v_1 , & v_2 \\ | & | \end{pmatrix} \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}$$

$$= \begin{pmatrix} | & | \\ e^{\lambda t} v_1 , & e^{\lambda t}(v_2 + t v_1) \\ | & | \end{pmatrix}$$

- In general: we need the knowledge about Jordan normal form for $A \in M_{n\times n}(\mathbb{R})$.

- $\exists$ Q $\mathbb{C}$-valued $n\times n$ invertible matrix s.t.

$$Q^{-1} A Q = \begin{pmatrix} \boxed{J_1} & 0 & \cdots & 0 \\ 0 & \boxed{J_2} & & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \cdots 0 & \boxed{J_k} \end{pmatrix} = J.$$

s.t. each $J_i = \begin{pmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix}$ is called a Jordan-block.

same eigenvalues for A.

Rk: Notice that different J_i's may associated with same eigenvalues.

Rk: • Let say we have a $k \times k$ -block $J_i = \left.k\right\{ \underbrace{\begin{pmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix}}_{k}$
with $\lambda_i \in \mathbb{C} \setminus \mathbb{R}$.

$\Longrightarrow$ $\exists$ another block $J_\ell = \begin{pmatrix} \bar{\lambda}_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \bar{\lambda}_i \end{pmatrix}$

• Let's consider $\begin{pmatrix} \boxed{J_1} & & \\ & \ddots & \\ & & \ddots \end{pmatrix}$ and $Q = \begin{pmatrix} | & & | & \\ v_1 & \cdots & v_k & \cdots \\ | & & | & \end{pmatrix}$

From $AQ = QJ$

$$\Longrightarrow Av_1 = \lambda_1 v_1, \quad Av_2 = \lambda_1 v_2 + v_1, \quad Av_3 = \lambda_1 v_3 + v_2$$

$$\dots \quad Av_k = \lambda_1 v_k + v_{k-1}$$

• if we have the block J_1 has real eigenvalues $\Longrightarrow$ we can take $v_1, \dots, v_k$ to be real.